Price of fairness for indivisible goods

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Research Seminar Information and Control – 05.05.2014
Resource allocation for indivisible items

Input

- a finite set of agents $N = \{1, \ldots, n\}$
Resource allocation for indivisible items

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- a finite set of indivisible items \( I = \{1, \ldots, m\} \)
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  Here: additive utilities \( u_i : 2^I \rightarrow \mathbb{R}_{\geq 0} \) or strict rankings
Resource allocation for indivisible items

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- a finite set of indivisible items $I = \{1, \ldots, m\}$
- preferences of the agents over the items
  Here: additive utilities $u_i : 2^I \rightarrow \mathbb{R}_{\geq 0}$ or strict rankings
- an optimization criterion
  Here: total utility $\sum_{i=1}^{n} u_i$ or cardinality of the assigned items
Resource allocation for indivisible items

Output

- the allocation $\bigcup_{i=1}^{n} A_i \subseteq I$ of all or a subset of the items to agents such that the optimization criterion is satisfied
Resource allocation for indivisible items

Output

- the allocation $\bigcup_{i=1}^{n} A_i \subseteq I$ of all or a subset of the items to agents such that the optimization criterion is satisfied.
Resource allocation for indivisible items

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What about stability?
... for the sake of stability and domestic harmony

<table>
<thead>
<tr>
<th>Fairness concepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ proportionality</td>
</tr>
<tr>
<td>▶ envy-freeness</td>
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<td>▶ equality</td>
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<tr>
<td>▶ ...</td>
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</tbody>
</table>
...for the sake of stability and domestic harmony

Fairness concepts

► proportionality
► envy-freeness
► equality
► ...

Here: envy-free allocations

Definition

An allocation $\bigcup_{i=1}^{n} A_i \subseteq I$ is envy-free if $u_i(A_i) \geq u_i(A_j)$ for any pair of agents $i, j$. 
The price of fairness

- let $A^*$ be the optimal allocation
- let $A^f$ be the optimal allocation satisfying the fairness criterion; 0 if no such allocation exists
- the price of fairness: $u(A^*) / u(A^f)$
The price of fairness for indivisible goods

The price of fairness

- let $A^*$ be the **optimal** allocation
- let $A^f$ be the **optimal** allocation satisfying the fairness criterion; 0 if no such allocation exists
- the price of fairness: $u(A^*) / u(A^f)$

**Research question**

Determine bounds for the price of fairness depending on some key parameters of allocation problems.
Part I: The price of envy-freeness for indivisible items with additive utility functions

Utility

- utility of a single item: $u_i : I \rightarrow \mathbb{R}_{\geq 0}$ for each agent $i \in N$
- additivity: $u_i(A_i) = \sum_{a \in A_i} u_i(A_i)$ for all $A_i \subseteq I$ and $i \in N$
- normalization: $u_i(I) = 1$ for all agents $i \in N$
- utility of an allocation: $u(A) = \sum_{i \in N} u_i(A_i)$ for all $A = (A_1, \ldots, A_n)$ with $\bigcup_{i=1}^n A_i = I$
Part I: The price of envy-freeness for indivisible items with additive utility functions

Utility

- utility of a single item: $u_i : I \rightarrow \mathbb{R}_{\geq 0}$ for each agent $i \in N$
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Price of envy-freeness

$p_{\text{envy}}(n)$: The supremum $u(A^*)/u(A^f)$ for all allocation problems with $n$ agents where $u(A^f) > 0$. 
Known results

**Lemma (Caragiannis et al., 2012)**

\[ p_{\text{envy}}(n) \leq n - \frac{1}{2} \quad \text{for } n \geq 2 \]

**Proof**

- an envy-free allocation is proportional, i.e., \( u_i(A_i) \geq \frac{1}{n} \Rightarrow u(A^f) \geq 1 \);
- if the optimal allocation \( A^* \) is envy-free, the price of envy-freeness is 1;
- if agent \( i \) is envy in \( A^* \), then \( u_i(A_i) < \frac{1}{2} \);
- if \( A^* \) is not envy-free, then \( u(A^*) < n - \frac{1}{2} \).
### Known results (cont.)

**Theorem (Caragiannis et al., 2012)**

\[ p_{\text{envy}}(n) \geq \frac{(3n + 7)}{9} - O\left(\frac{1}{n}\right) \]

**Proof**

A sophisticated construction using \( m \geq \frac{n^2}{4} + \frac{3n}{2} + 2 \) items.

**Corollary (Caragiannis et al., 2012)**

\[
\begin{align*}
1.5 &= \frac{3}{2} \leq p_{\text{envy}}(1) \leq \frac{1}{1} = 1.5 \\
1.75 &= \frac{7}{4} \leq p_{\text{envy}}(2) \leq \frac{3}{2} = 1.5 \\
1.9286 &\approx \frac{27}{14} \leq p_{\text{envy}}(3) \leq \frac{5}{2} = 2.5 \\
&\ldots \\
&\ldots \\
1.9286 &\approx \frac{27}{14} \leq p_{\text{envy}}(4) \leq \frac{7}{2} = 3.5
\end{align*}
\]
Known results (cont.)

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\end{align*}
\]
The price of envy-freeness in terms of the number of agents and items

**Definition**

\( p_{\text{envy}}(n, m) \): The supremum \( u(A^*)/u(A^f) \) for all allocation problems with \( n \) agents and \( m \) items, where \( u(A^f) > 0 \).
The price of envy-freeness in terms of the number of agents and items

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Observation

\( p_{\text{envy}}(n, m) \) is undefined for \( m < n \).

Remark

Using a matching problem it can be decided in polynomial time whether there exists an envy-free allocation for \( m = n \).
Bounds for $p_{\text{envy}}(n, m)$

**Observation**

- $p_{\text{envy}}(n, m) \leq p_{\text{envy}}(n, m + 1)$ for all $n, m \in \mathbb{N}$;
- $p_{\text{envy}}(n, m) \leq p_{\text{envy}}(n)$ for all $n, m \in \mathbb{N}$ and $p_{\text{envy}}(1, 1) = 1$

**Lemma**

$p_{\text{envy}}(2, 2) = 1$.

**Proof**

- each agent is assigned 1 item in any envy-free allocation;
- each agent obtains one of her most preferred items;

<table>
<thead>
<tr>
<th>$n/m$</th>
<th>1</th>
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<tr>
<td>1</td>
<td>$a_1 \geq \frac{1}{2}$</td>
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Determination of $p_{envy}(2, 3)$

**Lemma**

$$p_{envy}(2, 3) = \frac{3}{2}$$

**Proof**

- $p_{envy}(2, 3) \leq p_{envy}(2) = \frac{3}{2}$;
- for $\frac{1}{4} > \varepsilon > 0$ consider the following table:

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$u(A^*) = \frac{3}{2} - \varepsilon$
$u(A^f) = 1 + \varepsilon$
Determination of $p_{envy}(n, n)$

Observations

- each agent is assigned 1 item in any envy-free allocation;
- each agent obtains one of her most preferred items.
Determination of $p_{\text{envy}}(n, n)$

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Question
Can the problem to find a worst case example be formulated as an optimization problem?
## Determination of $p_{\text{envy}}(n, n)$

### Observations
- each agent is assigned 1 item in any envy-free allocation;
- each agent obtains one of her most preferred items.

### Question
Can the problem to find a worst case example be formulated as an optimization problem?

### Assumptions
Item $i$ is one of the most preferred items for agent $i$. 
An ILP formulation

\[
\max \sum_{i=1}^{n} \sum_{j=1}^{n} z_{i,j} - \sum_{i=1}^{n} x_{i,i} \\
\text{s.t.} \quad x_{i,j} \in \mathbb{R}_{\geq 0} \quad \forall \ 1 \leq i, j \leq n \quad \text{(utility)} \\
\sum_{j=1}^{n} x_{i,j} = 1 \quad \forall \ 1 \leq i \leq n \quad \text{(utility sums to 1)} \\
x_{i,i} \geq x_{i,j} \quad \forall \ 1 \leq i, j \leq n \quad \text{(envy-freeness)} \\
y_{i,j} \in \{0, 1\} \quad \forall \ 1 \leq i, j \leq n \quad \text{(optimal choice)} \\
\sum_{i=1}^{n} y_{i,j} = 1 \quad \forall \ 1 \leq j \leq m \quad \text{(chosen exactly once)} \\
z_{i,j} \in \mathbb{R}_{\geq 0} \quad \forall \ 1 \leq i, j \leq n \quad \text{(contribution to welfare)} \\
z_{i,j} \leq \min(y_{i,j}, x_{i,j}) \quad \forall \ 1 \leq i, j \leq n
\]
Lower bounds for $p_{envy}(n, n)$

**Lemma**

\[ p_{envy}(2k, 2k) \geq \frac{2k}{k+1} = 2 - \frac{2}{k+1} \quad \text{for } k \geq 1 \]

**Proof**

\[
\begin{array}{cccccccc}
1 & \frac{1}{2k} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{1}{2k} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
k & \frac{1}{2k} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{1}{2k} \\
k + 1 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \cdots & \cdots & 0 \\
k + 2 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2k & 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
\]

\[ u(A_f') = k \cdot \frac{1}{2} + k \cdot \frac{1}{2k} = \frac{k+1}{2} \]

\[ u(A^*) = k \cdot 2 \cdot \frac{1}{2} = k \]
Lower bounds for $p_{\text{envy}}(n, n)$ (cont.)

**Remark**
The previous bound is tight for $k = 1, 2, 3$.

**Lemma**

$$p_{\text{envy}}(2k + 1, 2k + 1) \geq \frac{2k}{k + 1} = 2 - \frac{2(4k - 1)}{6k^2 + 13k + 2} \quad \text{for} \quad k \geq 2$$

**Remark**
The previous bound is tight for $k = 2, 3$. 
Lower bounds for $p_{\text{envy}}(n, n)$ (cont.)

**Proof**

<table>
<thead>
<tr>
<th></th>
<th>$\frac{1}{2k+1}$</th>
<th>$\ldots$</th>
<th>$\ldots$</th>
<th>$\ldots$</th>
<th>$\ldots$</th>
<th>$\ldots$</th>
<th>$\frac{1}{2k+1}$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k+1$</td>
<td>$\frac{1}{2k+1}$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\frac{1}{2k+1}$</td>
</tr>
<tr>
<td>$k+2$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\ldots$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$2k$</td>
<td>0</td>
<td>$\ldots$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$2k+1$</td>
<td>0</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\ldots$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

$u(A^f) = \frac{k+1}{2k+1} + \frac{k-1}{2} + \frac{1}{3} = \frac{1}{6} \cdot \frac{6k^2 + 7k + 5}{2k+1}$

$u(A^*) = (k - 1) \cdot 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} = k$
Structure of the optimal solutions

row pattern: \( \left( \frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots, 0 \right) \) for \( 2 \leq k \leq n \rightarrow \frac{1}{k} \)

matrix description: \( \left( \frac{1}{2} v_2, \frac{1}{3} v_3, \ldots, \frac{1}{n} v_n \right) \) with \( \sum_{i=2}^{n} v_i = n \)

\[ u(A^f) = \sum_{i=2}^{n} v_i \cdot \frac{1}{i} \]

\( A^* = \left( \frac{1}{2} w_2, \frac{1}{3} w_3, \ldots, \frac{1}{n} w_n \right) \) with \( \sum_{i=2}^{n} w_i = n \) and \( w_i \leq i \cdot v_i \) for all \( 2 \leq i \leq n \)
An ILP formulation

\[
\max \sum_{i=2}^{n} \frac{w_i}{i} - \sum_{i=2}^{n} \frac{v_i}{i}
\]

\(v_i \in \mathbb{Z}_{\geq 0}\) \quad \forall 2 \leq i \leq n \quad \text{(matrix description)}

\[\sum_{i=2}^{n} v_i = n\] \quad \text{\(n\) agents, rows}

\(w_i \in \mathbb{Z}_{\geq 0}\) \quad \forall 2 \leq i \leq n \quad \text{(optimal choice)}

\[\sum_{i=2}^{n} w_i = n\] \quad \text{\(n\) items, columns}

\(w_i \leq i \cdot v_i\) \quad \forall 2 \leq i \leq n \quad \text{(number of occurrences)}
## Exact results

<table>
<thead>
<tr>
<th>$n$</th>
<th>$v$</th>
<th>$w$</th>
<th>$u(A^*)$</th>
<th>$u(A^f)$</th>
<th>$p_{\text{envy}}(n, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>($\frac{1}{2}, \frac{1}{3}$)</td>
<td>($\frac{1}{2}, \frac{1}{3}$)</td>
<td>$4$</td>
<td>$\frac{7}{3}$</td>
<td>$\frac{8}{7} \approx 1.1428$</td>
</tr>
<tr>
<td>4</td>
<td>($\frac{1}{2}, \frac{1}{3}$)</td>
<td>($\frac{1}{2}, \frac{1}{4}$)</td>
<td>$2$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{4}{3} \approx 1.3333$</td>
</tr>
<tr>
<td>5</td>
<td>($\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$)</td>
<td>($\frac{1}{2}, \frac{1}{3}$)</td>
<td>$2$</td>
<td>$\frac{43}{30}$</td>
<td>$\frac{60}{43} \approx 1.3953$</td>
</tr>
<tr>
<td>6</td>
<td>($\frac{1}{2}, \frac{1}{3}$)</td>
<td>($\frac{1}{2}, \frac{1}{6}$)</td>
<td>$3$</td>
<td>$2$</td>
<td>$\frac{3}{2} = 1.5$</td>
</tr>
<tr>
<td>7</td>
<td>($\frac{1}{2}, \frac{1}{3}, \frac{1}{7}$)</td>
<td>($\frac{1}{2}, \frac{1}{3}$)</td>
<td>$3$</td>
<td>$\frac{21}{40}$</td>
<td>$\frac{63}{40} = 1.575$</td>
</tr>
<tr>
<td>8</td>
<td>($\frac{1}{2}, \frac{1}{3}, \frac{1}{8}$)</td>
<td>($\frac{1}{2}, \frac{1}{3}$)</td>
<td>$3$</td>
<td>$\frac{43}{24}$</td>
<td>$\frac{72}{43} \approx 1.6744$</td>
</tr>
<tr>
<td>9</td>
<td>($\frac{1}{3}, \frac{1}{6}$)</td>
<td>($\frac{1}{3}$)</td>
<td>$3$</td>
<td>$\frac{5}{3}$</td>
<td>$\frac{9}{5} = 1.8$</td>
</tr>
<tr>
<td>50</td>
<td>($\frac{1}{7}, \frac{1}{8}, \frac{1}{50}$)</td>
<td>($\frac{1}{7}, \frac{1}{8}$)</td>
<td>$7$</td>
<td>$\frac{2579}{1400}$</td>
<td>$\geq \frac{9800}{2579} \approx 3.7999$</td>
</tr>
</tbody>
</table>
### Lower bounds for $p_{\text{envy}}(n, n)$ (cont.)

**Lemma**

$$p_{\text{envy}}(n, n) \geq \frac{1}{2} \cdot \sqrt{n}$$

**Proof**

For $v = \left(\frac{1}{k}, \frac{1}{n}^{n-k}\right)$ and $w = \left(\frac{1}{k}, \frac{1}{n}^{n-k^2}\right)$ with $k = \lfloor \sqrt{n} \rfloor$ we have

$$u(A^*) = k + \frac{n - k^2}{k} = \frac{n}{k} \geq \sqrt{n}$$

and

$$u(A^f) = 1 + \frac{n - k}{n} = 2 - \frac{k}{n} \leq 2 - \frac{\sqrt{n} - 1}{n} \leq 2.$$
Conjectures

Conjecture

\[ p_{\text{envy}}(n, n) \in \Theta(\sqrt{n}) \]

Conjecture

\[ p_{\text{envy}}(n, n) \] can be obtained using a configuration described by \( v \) with \( v_i = 0 \) for all \( i \notin \{ \lfloor \sqrt{n} \rfloor, \lceil \sqrt{n} \rceil, n \} \).

Conjecture

\[ p_{\text{envy}}(10, 10) = \frac{180}{97} \approx 1.8557 \]

\[ p_{\text{envy}}(11, 11) = \frac{198}{103} \approx 1.9223 \]

\[ \ldots \]
Part II: The price of envy-freeness for indivisible items with the cardinality of the assigned items as utility

### Utility

- **utility of an allocation**: $u(A) = \sum_{i \in N} |A_i|$ for all $A = (A_1, \ldots, A_n)$ with $\bigcup_{i=1}^{n} A_i \subseteq I$

### Price of envy-freeness ($\rho_{envy}$)

$\rho_{envy}(n)$ ($\rho_{envy}(n, m)$): The supremum $u(A^*)/u(A^f)$ for all allocation problems with $n$ agents (and $m$ items) where $u(A^f) > 0$. 
The worst case example

Observation

If $u(A^f) > 0$, then $u(A^f) \geq n$ and $m \geq n$. $u(A^*) = m$. 
## The worst case example

### Observation

If $u(A^f) > 0$, then $u(A^f) \geq n$ and $m \geq n$. $u(A^*) = m$.

### Lemma

$$p_{envy}(n, m) = \frac{m}{n}$$

### Proof

Let $\pi = (m - n + 1, \ldots, n)$ be a permutation of the last $n$ items. If the preference ordering for player $i$ is given by $[1, \ldots, m - n, \pi^i]$, then we have $u(A^f) = n$. 
Observation
For two players consider the preferences \((1, 2, 3 \ldots, m)\) and \((2, 3, \ldots, m, 1)\), where the number of items \(m\) is even.

\[ BT^a \rightarrow 2 \quad AL^a \rightarrow m \]

Technical details: \(S_A\): odd numbers, \(S_B\): even numbers, 
\(f_A : 1 \rightarrow 2, 3 \rightarrow 4, \ldots, n - 1 \rightarrow n, \) and 
\(f_B : 2 \rightarrow 3, 4 \rightarrow 5, \ldots, n - 2 \rightarrow n - 1, n \rightarrow 1. \)

\(^a\)You have to wait for Marc’s talk.
Open problems

- find a tighter upper bound for $p_{\text{envy}}(n)$;
- determine some more exact values of $p_{\text{envy}}(n, m)$;
- prove $p_{\text{envy}}(n, n) \in \Theta(\sqrt{n})$;
- \ldots (What would be nice to know?)

Thank you for your attention